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Correction of Cosmological Expansion to Angular Deflection of Light and Radar Echo Delay

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Abstract The angular deflection of light and radar echo delay are famous results predicted by general relativity. The gravitational lensing problems depend on the deviation of light from its straight line path in its basic equation. Using the Robertson-McVittie spacetime metric, which coincides thoroughly with the Schwarzschild metric in the isotropic coordinate and the FLRW metric for curvature parameter $k = 0$ when $M = 0$, we discuss the correction of cosmological expansion to the angular deviation of light path and the radar echo delay. The deviation terms arising from the expansion of universe are found to be simply $-\frac{4GM}{r_{min}c^2}(\frac{H_0^2}{2c^2}r_{min}^2)$ for angular deviation and $\frac{2H_0^2}{3c^3}(r_A^3 + r_B^3)$ for radar echo delay.

Keywords Robertson-McVittie · Deviation of light · Radar echo delay

1 Introduction

The first discovery of multiple-image system(twin quasistellar objects) in 1979 [1] made the field of gravitational lensing, which was firstly predicted by Einstein [2], become one of the most active subjects of astrophysical research [3–7]. This field has arisen great interests among astronomers and astrophysicists to study gravitational lensing problems (see M. Bartelmann and P. Schneider for review [8]), especially after the discovery of the accelerating expansion of our universe, It is used as an effective tool to study the properties of dark energy [9–11]. However, in the original paper of gravitational lensing, the deflection angle of light propagation is obtained in the Schwarzschild metric, which is most general static, spherically symmetric, vacuum solution of Einstein's field equations. The result does not include the influence of other possible source. Recently, there are many discussion [12–15, 17, 18] in the Roberson-McVittie spacetime [16], which describe the spacetime around the point

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mass singularity embedded in the FLRW background metric. In this paper, we will consider the angular deviation of light propagation and the radar echo delay in this spacetime metric, where we can obtain the correction originating from the effect of the cosmological expansion. Even though the corrections we obtained in this paper compared to the classical results are too small to be detected at present experiment level, we still think the discussion are at least theoretically interesting since it can relate the two hot spots: the gravitational lensing and the accelerating expansion of the universe.

The spacetime around the point mass singularity embedded in the FLRW background metric can be described by Robertson-McVittie metric. The standard form of Robertson-McVittie metric is given by [12–18]:

$$ds^2 = \left[\frac{1-\frac{GM}{2c^2ra(t)}}{1+\frac{GM}{2c^2ra(t)}}\right]^2 dt^2 - \left[1+\frac{GM}{2c^2ra(t)}\right]^4 a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{1}$$

where G is the gravitational constant, M is the mass of central body, c is the speed of light in vacuum, and $a(t)$ is a scale factor.

Since the observational models in the solar system are formulated not in the comoving coordinate system as (1) but in the barycentric celestial reference system (BCRS) [15], it is more adequate to take the reference system as close as possible to BCRS. converting (1) into the nearly proper coordinate system, we finally get the following form as a working spacetime metric [15]:

$$\begin{aligned} ds^2 &= \left(1-\frac{2GM}{c^2r}-\frac{H_0^2}{c^2}r^2\right)c^2dt^2 - \left(1+\frac{2GM}{c^2r}+\frac{H_0^2}{c^2}r^2\right)dr^2 \\ &\quad - r^2(d\theta^2+\sin^2\theta d\varphi^2) \\ &= \left(1-\frac{r_g}{g}-\frac{H_0^2}{c^2}r^2\right)c^2dt^2 - \left(1+\frac{r_g}{g}+\frac{H_0^2}{c^2}r^2\right)dr^2 - r^2(d\theta^2+\sin^2\theta d\varphi^2) \end{aligned} \tag{2}$$

where $r_g = \frac{2GM}{c^2}$, being the Schwarzschild radius.

The lagrangian equation for light is

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 \tag{3}$$

where $x^\mu = x^\mu(t, r, \theta, \varphi)$ and Lagrangian L being:

$$L(x^\mu, \dot{x}^\mu) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\mu = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2. \tag{4}$$

Substituting (2) to (4), we get:

$$L = \frac{1}{2}\left[\left(1-\frac{r_g}{r}-\frac{H_0^2}{c^2}r^2\right)c^2\dot{t}^2 - \left(1+\frac{2GM}{c^2r}+\frac{H_0^2}{c^2}r^2\right)\dot{r}^2 - r^2(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)\right] \tag{5}$$

where the dot symbol denotes $d/d\tau$. Equation (5) does not contain φ and t explicitly, so we can deduce following equations from (3) and (5):

$$\frac{\partial L}{\partial \dot{\varphi}} = -r^2\sin^2\theta\dot{\varphi} = -\frac{1}{A}, \tag{6}$$

$$\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{r_g}{r} - \frac{H_0^2}{c^2} r^2\right)\dot{t} = B. \tag{7}$$

Where A and B are integral constants. Equations (6) and (7) describe the conservation of angular momentum and energy respectively.

It is well known that the motion of a particle in spherical symmetric gravitational field is confined to a plane, so we can choose the z axis to be perpendicular to this plane. Then we have $\theta = \pi/2$, $\dot{\theta} = 0$ and (6) become $r^2\dot{\varphi} = 1/A$. For photon $ds^2 = 0$, from (4) and (5) we can obtain:

$$\left(1 - \frac{r_g}{r} - \frac{H_0^2}{c^2} r^2\right)\dot{t}^2 - \left(1 + \frac{r_g}{g} + \frac{H_0^2}{c^2} r^2\right)\dot{r}^2 - r^2\dot{\varphi}^2 = 0. \tag{8}$$

2 Angular Deflection of Light Path

To get the angular deflection of light path in gravitational field, we have to deduce the equation relating r and φ. This equation can be obtained by eliminating t and τ from (7), (8):

$$\frac{1 - 2\alpha\beta - \alpha^2 - \beta^2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + 1 = A^2B^2r^2 + \alpha + \beta \tag{9}$$

where we set $\alpha = \frac{r_g}{r} = \frac{2GM}{c^2 r}$, $\beta = \frac{H_0^2}{c^2} r^2$ for convenience, but they are not constant.

If the mass M of the intervening stellar system is zero and we do not consider the contribution of Hubble parameter which originates from the cosmological expansion, then both α and β are zero and (9) becomes:

$$\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + 1 = A^2B^2r^2 \tag{10}$$

with the solution being

$$r = \frac{r_{min}}{\cos\varphi}. \tag{11}$$

Where $r_{min} = 1/(AB)$. If $M \neq 0$ and considering the contribution of Hubble parameter H_0, the solution of (9) can be expressed as [19, 20]:

$$r = \frac{r_{min}}{\cos\varphi + \delta(\varphi)}. \tag{12}$$

Where δ is a tiny value, represents the correction due to gravitational field and the expansion of universe. Substitute (12) into (9), neglecting the second order of α, β and the terms with order equals to and greater than $\delta(\varphi)^2$ (since all of them are all very small, we can understand this later), we arrive at the following differential equation for $\delta(\varphi)$:

$$\frac{d\delta}{d\varphi}\sin\varphi - \delta\cos\varphi + \frac{GM\cos^3\varphi}{r_{min}c^2} = -\frac{H_0^2}{2c^2} r_{min}^2. \tag{13}$$

Integrating (13), we obtained following exact solution:

$$\delta(\varphi) = \frac{2GM}{r_{min}c^2} + \frac{H_0^2}{2c^2} r_{min}^2 \cos\varphi - \frac{GM}{r_{min}c^2}\cos^2\varphi + C_1\sin\varphi. \tag{14}$$

We will set the integral constant C_1 to be zero for next calculation since $\delta(\varphi) = 0$ when $M = 0$ and $H_0 = 0$. We obtain the orbit of light from (12) and (14):

$$r = \frac{r_{min}}{-\frac{GM}{r_{min}c^2}\cos^2\varphi + (1 + \frac{H_0^2}{2c^2}r_{min}^2)\cos\varphi + \frac{2GM}{r_{min}c^2}}. \tag{15}$$

The direction of the asymptote is described by:

$$-\frac{GM}{r_{min}c^2}\cos^2\varphi + \left(1 + \frac{H_0^2}{2c^2}r_{min}^2\right)\cos\varphi + \frac{2GM}{r_{min}c^2} = 0. \tag{16}$$

There are two solutions for (16):

$$\cos\varphi = \frac{\beta' + 1}{2\alpha'}\left[1 \pm \sqrt{1 + 8\left(\frac{\alpha'}{\beta' + 1}\right)^2}\right]. \tag{17}$$

Where $\alpha' = \frac{GM}{r_{min}c^2}$, $\beta' = \frac{H_0^2}{2c^2}r_{min}^2$. Considering a typical stellar system like our own Galaxy with $M = 10^{41}$ kg and $r_{min} = 10^{20}$ m, we can estimate that $\alpha' \sim 10^{-6}$ and $\beta' \sim 10^{-13}$. Then (17) became:

$$\cos\varphi = \frac{\beta' + 1}{2\alpha'}\left[1 \pm \left(1 + 4\left(\frac{\alpha'}{\beta' + 1}\right)^2\right)\right] = \begin{cases} \frac{\beta'+1}{\alpha'} \\ -\frac{2\alpha'}{\beta'+1} \end{cases}. \tag{18}$$

The solution $\cos\varphi = (\beta' + 1)/\alpha'$ should be thrown away since $|\cos\varphi| \le 1$. So the solution of (16) is $\cos\varphi = -2\alpha'/(\beta' + 1)$, just the solution if we omit the first term of (16). It tells us that $\cos\varphi$ is a very tiny value. The angular coordinate φ of the asymptote for the light orbit can be expressed:

$$\varphi = \frac{\pi}{2} + \frac{1}{2}\triangle\varphi_N. \tag{19}$$

Where $\triangle\varphi_N$ is the deflection angle of light. Taking cosine on both sides of (19) and the approximation of $\sin(\triangle\varphi_N/2) \sim \triangle\varphi_N/2$, we obtain:

$$\triangle\varphi_N = \frac{4\alpha'}{\beta' + 1} = \frac{4GM}{r_{min}c^2(1 + \frac{H_0^2}{2c^2}r_{min}^2)} \simeq \frac{4GM}{r_{min}c^2}\left(1 - \frac{H_0^2}{2c^2}r_{min}^2\right). \tag{20}$$

The first term on the right is just the result of light deflection obtained in Schwarzschild metric due to the effect of original general relativity, while the term associated with $\frac{H_0^2}{2c^2}r_{min}^2$ described the correction from the cosmological expansion. The relative factor of the correction of cosmological expansion is $\frac{H_0^2}{2c^2}r_{min}^2$, being order of $\sim 10^{-13}$. Clearly, up to the experimental resolution nowadays, we need not consider this correction associated with H_0. However, academic estimation of the contribution is interesting because the expansion of universe is really an existence. We now know that we do not need to consider the expansion of the universe and can safely use the result of angular deflection to gravitational lensing. Another interesting result is that the contribution of the expansion of universe decreases the deflection angle since the second term on the right in (20) has a minus sign.

3 Time Delay of Radar Echo

In the last section, the key problem lies on finding the direction of light propagation, i.e. we can only figure out the relation between r and φ, do not consider the time element. If we consider a radar signal sent from planet P_A to planet P_B and then the reflected echo is received at P_A, we can obtain the time delay in this case. If there is no gravitational field, the time equals $2D_{AB}/c$, where D_{AB} is the straight distance between P_A and P_B. If there are gravitational field existing around the propagation path, the real propagation distance will be larger and leads to a time delay of the echo. Moreover, if we take the metric equation (2), i.e. including the effect of the cosmological expansion, there should be a correction to time of delay. We will try to find the correction in this section.

We know that $ds^2 = 0$ for the light signal. We still have $\theta = \pi/2$ since we can confine the motion of light to be in a plane. Then we get following equation from (2):

$$(1-\alpha-\beta) - \frac{1}{c^2}\left[(1+\alpha+\beta)\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] = 0. \tag{21}$$

We can obtain following equation from (6) and (7):

$$\frac{r^2\frac{d\varphi}{d\tau}}{[(1-\alpha-\beta)\frac{dt}{d\tau}]} = \frac{r^2\frac{d\varphi}{dt}}{(1-\alpha-\beta)} = \frac{1}{AB} = F. \tag{22}$$

Substituting (22) into (21), we get

$$(1-\alpha-\beta) - \frac{1}{c^2}\left[(1+\alpha+\beta)\left(\frac{dr}{dt}\right)^2 + \frac{F^2}{r^2}(1-\alpha-\beta)^2\right] = 0. \tag{23}$$

Namely:

$$\left(1-\frac{2GM}{c^2r}-\frac{H_0^2}{c^2}r^2\right) - \frac{1}{c^2}\left[\left(1+\frac{2GM}{c^2r}+\frac{H_0^2}{c^2}r^2\right)\left(\frac{dr}{dt}\right)^2 + \frac{F^2}{r^2}\left(1-\frac{2GM}{c^2r}-\frac{H_0^2}{c^2}r^2\right)^2\right] = 0.$$

When r equals r_{min}, the closest point between the signal and O, $dr/dt = 0$. Then we obtain the explicit expression for F:

$$F^2 = r_{min}^2 c^2 \Big/ \left(1-\frac{2GM}{c^2 r_{min}}-\frac{H_0^2}{c^2}r_{min}^2\right). \tag{24}$$

Now we can obtain dr/dt from (23) and (24):

$$\left(\frac{dr}{dt}\right)^2 = c^2\frac{(1-\alpha-\beta)}{(1+\alpha+\beta)}\left[1-\frac{(1-\alpha-\beta)}{(1-\alpha'-\beta')}\Big/\left(\frac{r^2}{r_{min}^2}\right)^2\right]. \tag{25}$$

Where $\alpha' = \alpha' = \frac{GM}{r_{min}c^2}$, $\beta' = \beta' = \frac{H_0^2}{2c^2}r_{min}^2$. We can take the approximation $1/(1+\alpha+\beta) \simeq (1-\alpha-\beta)$ and get following equation from (25):

$$\frac{dr}{dt} = c(1-\alpha-\beta)\sqrt{1-\frac{(1-\alpha-\beta)}{(1-\alpha'-\beta')}\Big/\left(\frac{r^2}{r_{min}^2}\right)^2}$$

$$= c\left(1 - \frac{2GM}{c^2 r} - \frac{H_0^2}{c^2} r^2\right)\sqrt{1 - \frac{r_{min}^2 (1 - \frac{2GM}{c^2 r} - \frac{H_0^2}{c^2} r^2)}{r^2 (1 - \frac{2GM}{c^2 r_{min}} - \frac{H_0^2}{c^2} r_{min}^2)}}. \tag{26}$$

Then we can integrate the coordinate time from above equation:

$$t(r, r_{min}) = \frac{1}{c}\int_{r_{min}}^{r} \frac{dr}{(1 - \frac{2GM}{c^2 r} - \frac{H_0^2}{c^2} r^2)\sqrt{1 - \frac{r_{min}^2 (1 - \frac{2GM}{c^2 r} - \frac{H_0^2}{c^2} r^2)}{r^2 (1 - \frac{2GM}{c^2 r_{min}} - \frac{H_0^2}{c^2} r_{min}^2)}}}. \tag{27}$$

To get the proper time, we should multiply the coordinate time by a factor of $(1 - \frac{2GM}{c^2 r} - \frac{H_0^2}{c^2} r^2)^{1/2}$, which is different from one by only 10^{-7}. So in the following study we take such a factor to be 1. Then the time from emitting a light signal to receiving its echo is:

$$T = 2[t(r_{P_A}, r_{min}) + t(r_{P_B}, r_{min})]. \tag{28}$$

Here r_{P_A}, r_{P_B} are spatial coordinates of the two planets with respect to the massive gravitational source O. In the flat space–time without gravitational source, the time for the signal to propagate from coordinate r to r_{min} is simply $t^0(r, r_{min}) = (r^2 - r_{min}^2)^{1/2}/c$.

Expanding the denominator to first order of $r_g/r = \frac{2GM}{c^2 r}$ and $\frac{H_0^2}{c^2}$, we have

$$t(r, r_{min}) \simeq \frac{1}{c}\int_{r_{min}}^{r} \frac{r\,dr}{\sqrt{r^2 - r_{min}^2}}\left[1 + \frac{r_g}{r} + \frac{r_g r_{min}}{2r(r + r_{min})}\right]$$
$$+ \frac{1}{c}\int_{r_{min}}^{r} \frac{r\,dr}{\sqrt{r^2 - r_{min}^2}}\left[\frac{H_0^2}{c^2} r^2 - \frac{H_0^2}{2c^2} r_{min}^2\right] \tag{29}$$

which gives the explicit integral expression

$$t(r, r_{min}) = \frac{1}{c}\left[\sqrt{r^2 - r_{min}^2} + r_g \ln\left(\frac{r + \sqrt{r^2 - r_{min}^2}}{r_{min}}\right) + \frac{r_g}{2}\sqrt{\frac{r - r_{min}}{r + r_{min}}}\right]$$
$$+ \frac{H_0^2}{6c^3}\sqrt{r^2 - r_{min}^2}(2r^2 + r_{min}^2). \tag{30}$$

The last term is associated with the correction arising from the expansion of universe while the other terms represent the correction to echo delay due to the massive stellar gravitational system.

The total echo delay then equals the time in (28) minus the one in flat space-time:

$$\triangle T \equiv T - \frac{2}{c}\sqrt{r_A^2 - r_{min}^2} - \frac{2}{c}\sqrt{r_B^2 - r_{min}^2}. \tag{31}$$

Since we have the condition $r_A, r_B \gg r_{min}$, then (31) can actually be simplified to

$$\triangle T \simeq \triangle T_0 + \triangle T_{H_0} \tag{32}$$

where

$$T_0 = \frac{4GM}{c^3}\left[\ln\left(\frac{4r_A r_B}{r_{min}^2}\right) + 1\right], \qquad \triangle T_{H_0} = \frac{2H_0^2}{3c^3}(r_A^3 + r_B^3).$$

When we do not consider the contribution of cosmological expansion, the classical result is retrieved. Suppose M represents the mass of the sun, P_A and P_B are the earth and the mercury respectively, the correction due to the expansion of universe is very small, being 10^{-27} s. So we also do not need to include such correction because our present experimental resolution can not reach the level to detect this tiny value. It is also very difficult to confirm this tiny value just being from the contribution of the expansion of our universe. So what we discuss and the results we obtain in this paper are purely academic. However we now know that, it is unnecessary to consider the expansion of the universe in the metric and the basic equation in gravitational lensing is reliable.

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